A Kinematic Conservation Law in Free Surface Flow

Sergey Gavrilyuk*
Aix Marseille Université, CNRS UMR 7343, IUSTI, 5 rue E. Fermi
Marseille 13013, France, and Novosibirsk State University, Novosibirsk 630090, Russia

Henrik Kalisch† and Zahra Khorsand‡
Department of Mathematics, University of Bergen
Postbox 7800, 5020 Bergen, Norway

January 20, 2015

Abstract

The Green-Naghdi system is used to model highly nonlinear weakly dispersive waves propagating at the surface of a shallow layer of a perfect fluid. The system has three associated conservation laws which describe the conservation of mass, momentum, and energy due to the surface wave motion. In addition, the system features a fourth conservation law which is the main focus of this note. It will be shown how this fourth conservation law can be interpreted in terms of a concrete kinematic quantity connected to the evolution of the tangent velocity at the free surface. The equation for the tangent velocity is first derived for the full Euler equations in both two and three dimensional flows, and in both cases, it gives rise to an approximate balance law in the Green-Naghdi theory which turns out to be identical to the fourth conservation law for this system. An analogous equation valid along each contact surface in the fluid bulk (and not only on the free surface) was also derived in both two and three dimensions.

1 Introduction

The surface water wave problem considered here concerns the motion of an inviscid incompressible fluid with a free surface and over an impenetrable rigid bottom. The mathematical description of this motion involves the Euler equations for perfect fluids coupled with free-surface boundary conditions. Solving these equations presents a rather complex mathematical problem, and in many instances, it can be expedient to use an approximate model for the description of the free surface and the motion of the fluid below. The classical simplifications of this problem are the linear theory which may be used for waves of small amplitude, and the shallow water theory, which may be used for long waves. A third class of approximations can be found in the so-called Boussinesq scaling which requires a certain relationship between the amplitude and wavelength of the waves to be described.

The Green-Naghdi system was derived as a long-wave model for surface water waves which are long, but which may not necessarily have small amplitude. In fact, the Green-Naghdi

*sergey.gavrilyuk@univ-amu.fr
†henrik.kalisch@math.uib.no
‡zahra.khorsand@math.uib.no

system was first found in the special case of one-dimensional waves over a flat bottom [40], in which case it can be thought of as a higher-order nonlinear shallow-water system. The system was later extended to two-dimensional surface waves [41], and put into the general context of fluid sheets by [22, 23]. If l is a characteristic wavelength and h_0 is the the mean water depth, we define the dimensionless small parameter $\beta = h_0^2/l^2$. The Green-Naghdi equations are obtained by depth-averaging the Euler system and keeping in the resulting set of equations only first order terms in β; without making any assumptions on the amplitude of the waves.

While the studies mentioned so far give a formal justification of the Green-Naghdi system as a wave model accommodating larger amplitude waves, a mathematical justification of this property was given for the Green-Naghdi and some related systems in [29, 30, 33, 34, 39]. Recent years have seen increased activity in both the study of modeling properties of the Green-Naghdi system ([4, 5, 8, 12, 18, 20, 21, 27, 32]) and in the development of numerical discretization techniques, such as in [17, 19, 31, 35].

The Green-Naghdi system is usually written in terms of the average horizontal velocity $\bar{u}(t,x)$ and the total flow depth $h(t,x) = h_0 + \eta(t,x)$, and take the form

$$h_t + (h\bar{u})_x = 0, \tag{1}$$

$$\bar{u}_t + \bar{u}\bar{u}_x + gh_x - \frac{1}{3h}\frac{\partial}{\partial x}\Big\{h^3\big(\bar{u}_{xt} + \bar{u}\bar{u}_{xx} - \bar{u}_x^2\big)\Big\} = 0. \tag{2}$$

Note that if higher-order terms in (2) are discarded, the system reduces to the classical shallow-water system. On the other hand, considering waves of infinitesimally small amplitude, one obtains the linearization of the Green-Naghdi system, which is identical to the linearized classical Boussinesq system [36].

Smooth solutions of (1), (2) also satisfy the following conservation laws:

$$\frac{\partial}{\partial t}\{h\} + \frac{\partial}{\partial x}\{h\bar{u}\} = 0, \tag{3}$$

$$\frac{\partial}{\partial t}\{h\bar{u}\} + \frac{\partial}{\partial x}\Big\{\frac{1}{2}gh^2 + h\bar{u}^2 - \frac{1}{3}h^3\bar{u}_{xt} + \frac{1}{3}h^3\bar{u}_x^2 - \frac{1}{3}h^3\bar{u}\bar{u}_{xx}\Big\} = 0, \tag{4}$$

$$\frac{\partial}{\partial t}\Big\{\frac{1}{2}h\left(gh + \bar{u}^2 + \tfrac{1}{3}h^2\bar{u}_x^2\right)\Big\} + \frac{\partial}{\partial x}\Big\{h\bar{u}\Big(gh + \tfrac{1}{2}\bar{u}^2 + \tfrac{1}{2}h^2\bar{u}_x^2 - \tfrac{1}{3}h^2(\bar{u}_{xt} + \bar{u}\bar{u}_{xx})\Big)\Big\} = 0, \tag{5}$$

$$\frac{\partial}{\partial t}\Big\{\bar{u} - hh_x\bar{u}_x - \frac{1}{3}h^2\bar{u}_{xx}\Big\} + \frac{\partial}{\partial x}\Big\{gh + \frac{1}{2}\bar{u}^2 - hh_x\bar{u}\bar{u}_x - \tfrac{1}{2}h^2\bar{u}_x^2 - \tfrac{1}{3}h^2\bar{u}\bar{u}_{xx}\Big\} = 0. \tag{6}$$

The conservation law (3) is easily seen to describe mass conservation. It can also be shown that (4) represents momentum conservation and (5) represents the energy conservation in the Green-Naghdi approximation [26]. The last conservation law (6) is not easily interpreted. Some authors have ascribed the conservation of angular momentum to this balance equation [12]. On the other hand, it was shown in [37] that it originates from the particle relabeling symmetry for a corresponding approximate Lagrangian of the Euler equations. The invariance of the integral of the density $\bar{u} - hh_x\bar{u}_x - \frac{1}{3}h^2\bar{u}_{xx} = \bar{u} + \frac{1}{3h}\frac{\partial}{\partial x}\left(h^2\frac{Dh}{Dt}\right)$, where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}$ over the real line has also been linked to the irrotationality assumption in the context of two-layer flow [16].

In the present work, the focus will be on giving a precise physical meaning to the quantities appearing in the fourth conservation law. As will come to light, the density appearing in the conservation law (6) is proportional to the tangential fluid velocity at the free interface, while the flux can be interpreted as an approximation of the energy per unit mass of fluid. The conservation law derives from a corresponding balance law which holds exactly for the surface water-wave problem for the full Euler equations, and can be written in Lagrangian variables as

$$\frac{\partial\left(\mathbf{u}\cdot\frac{\partial\mathbf{x}}{\partial s}\right)}{\partial t}+\frac{\partial}{\partial s}\Big(\frac{p}{\rho}+gh-\frac{|\mathbf{u}|^2}{2}\Big)=0. \tag{7}$$

Here $\mathbf{u}$ denotes the fluid velocity, $\mathbf{x}(s,t)$ is the fluid particle motion, p is the pressure, and ρ is the density. The variable s featuring in this balance law denotes a parametrization of an arc lying entirely in the fluid domain or in the free surface. In the particular case where the arc is located in the free surface, the term $\mathbf{u}\cdot\frac{\partial\mathbf{x}}{\partial s}$ can be interpreted as the tangent velocity along the free surface, multiplied by the local element of arclength. The balance law (7) provides the basis for the conservation law (6) in the Green-Naghdi approximation. This conservation law actually holds even for the three-dimensional water-wave problem, but in this case, an additional term enters the corresponding identity in the Green-Naghdi approximation, so that the identity corresponding to (6) in three dimensions is not a pure conservation law.

The disposition of the present paper is as follows. In the next section, we describe how to obtain the balance equation (7) in the context of the full Euler equations. Then, the Green-Naghdi approximation is considered, and the associate approximate density and flux are obtained using a method introduced in [1]. In Section 4, the three-dimensional case is considered, and Section 5 contains a brief discussion.

2 Kinematic balance laws in two-dimensional flows

2.1 2D balance law for perfect fluids

The motion of a homogeneous inviscid and incompressible fluid with a free surface over a flat bottom can be described by the Euler equations with appropriate boundary conditions. The unknowns are the surface elevation $\eta(t,x)$, and the horizontal and vertical fluid velocity components $u(t,x,z)$ and $w(t,x,z)$, respectively. The problem may be posed on a domain $\Omega_t=\{(x,z)|0<z<h_0+\eta(t,x)\}$ which extends to infinity in the positive and negative x-direction. The two-dimensional Euler equations are

$$\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\frac{1}{\rho}\nabla p=\ \mathbf{g},\qquad \text{in } \Omega_t, \tag{8}$$

$$\nabla\cdot\mathbf{u}=\ 0,\qquad \text{in } \Omega_t, \tag{9}$$

where $\mathbf{u}=(u,w)^T$ represents the velocity field, and $\mathbf{g}=(0,-g)^T$ is the gravitational acceleration. The free-surface boundary conditions are given by requiring the pressure to be equal to atmospheric pressure at the surface, i.e. $p=p_{atm}$ if surface tension effects are neglected, and the kinematic boundary condition

$$\eta_t+u\eta_x=w$$

for $z=h_0+\eta(t,x)$. For the purposes of this paper, we work with the gauge pressure, so that we may take $p_{atm}=0$.

Following Zakharov [42], we introduce semi-Lagrangian coordinates (x,λ). The change of variables $(x,z)\to(x,\lambda)$ is given by the function

$$z=\Phi\left(t,x,\lambda\right),$$

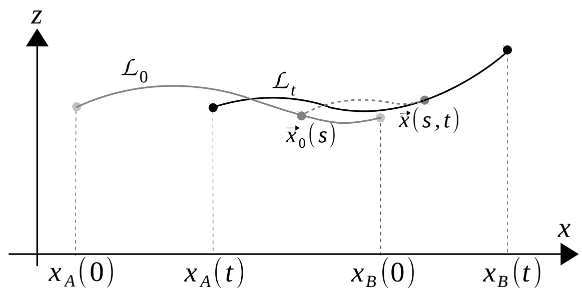

Figure 1: This figure indicates the time evolution of an arc $\mathcal{L}_0$ which lies entirely in the free surface.

where $\Phi\,(t, x, \lambda)$ is the solution of the equation

$$\frac{\partial \Phi}{\partial t} + u\frac{\partial \Phi}{\partial x} = w,$$

with the initial condition

$$\Phi\big|_{t=0} = \lambda h_0 + \lambda \eta_0(x),$$

where $\eta_0(x)$ is the initial disturbance of the free surface. At each time t, this change of variables represents a foliation of the domain in (x, z) variables by contact lines indexed by $0 \le \lambda \le 1$, and where each line corresponds to $\lambda = const$ in the (x, λ)-plane. The free surface $z = h_0 + \eta(t, x)$ corresponds to the value $\lambda = 1$. It can be shown that in semi-Lagrangian coordinates, the material derivative is defined only in terms of the horizontal velocity, and the curvilinear domain occupied by the fluid in the plane $\{(x, z), 0 < z < h_0 + \eta(t, x)\}$ maps to a strip in the plane $\{(x, \lambda), 0 < \lambda < 1\}$. Let $\mathcal{L}_t^\lambda$ be a finite material arc lying entirely in one of the contact lines defined above, starting at a point $\mathbf{x}_A(t) = (x_A(t), z_A(t))^T$, and ending at another point $\mathbf{x}_B(t) = (x_B(t), z_B(t))^T$. The arc corresponding to the value $\lambda = 1$ lies entirely in the free surface, and will be denoted simply by $\mathcal{L}_t$.

Parametrizing the initial arc $\mathcal{L}_0^\lambda$ using the real parameter s, we obtain the description of the arc $\mathcal{L}_t^\lambda$ in the form

$$\mathbf{x} = \boldsymbol{\varphi}(t, s, \lambda) = \left(\varphi^x\,(t, s, \lambda)\,, \varphi^z\,(t, s, \lambda)\right)^T, \quad s_A \le s \le s_B,$$

where $\mathbf{x} = \boldsymbol{\varphi}(t, s, \lambda)$ is the solution of the Cauchy problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(t, \mathbf{x}), \qquad \mathbf{x}|_{t=0} = \mathbf{x}_0(s, \lambda), \qquad \mathbf{x}_0(s, \lambda) \in \mathcal{L}_0^\lambda.$$

Using the Euler equations (8), (9) one can obtain the time evolution of the total drift γ^λ along $\mathcal{L}_t^\lambda$ in the form

$$\frac{d\gamma^\lambda}{dt} = \frac{d}{dt}\int\limits_{\mathcal{L}_t^\lambda} \mathbf{u}\cdot d\mathbf{x} = \left(\frac{|\mathbf{u}|^2}{2} - \frac{p}{\rho} - gz\right)\Bigg|_{\mathbf{x}_A(t)}^{\mathbf{x}_B(t)}. \tag{10}$$

Observing that

$$\int\limits_{\mathcal{L}_t^\lambda} \mathbf{u}{\cdot}d\mathbf{x} = \int\limits_{\mathcal{L}_0^\lambda} \mathbf{u} \cdot \frac{\partial \mathbf{x}}{\partial s}\, ds,$$

we see that (10) can also be written in local form as (7) along the material curve $\mathcal{L}_t^\lambda$. In fact, the previous computation is nothing but the standard proof of Kelvin's circulation theorem, but applied on a contour which is not necessarily closed. Taking a contour embedded in the material line $\mathcal{L}_t^\lambda$, we obtain from (10) the conservation law in the Lagrangian coordinates

$$\frac{\partial\left(\mathbf{u} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial s}\right)}{\partial t} + \frac{\partial}{\partial s}\left(\frac{p}{\rho} + gz - \frac{|\mathbf{u}|^2}{2}\right) = 0. \tag{11}$$

Introducing $K^\lambda(t,x)$ defined along $\mathcal{L}_t^\lambda$ by

$$\mathbf{u}(t,s,\lambda){\cdot}\frac{\partial \boldsymbol{\varphi}(t,s,\lambda)}{\partial s} = K^\lambda(t,x)\frac{\partial \varphi^x(t,s,\lambda)}{\partial s}, \tag{12}$$

we obtain the following conservation law in the Eulerian coordinates (t,x) along $\mathcal{L}_t^\lambda$:

$$\frac{\partial K^\lambda}{\partial t} + \frac{\partial}{\partial x}\left(u\left(t,x,\lambda\right)K^\lambda + \frac{p\left(t,x,\lambda\right)}{\rho} + gz\left(t,x,\lambda\right) - \frac{1}{2}(u^2\left(t,x,\lambda\right) + w^2\left(t,x,\lambda\right))\right) = 0 \tag{13}$$

In order to understand the meaning of K^λ, we let $\mathbf{n}^\lambda = \left(-\frac{\partial z(t,x,\lambda)}{\partial x}, 1\right)^T$ be the unit normal to $\mathcal{L}_t^\lambda$, scaled by the element of arclength of $\mathcal{L}_t^\lambda$. By the same token, let $\boldsymbol{\tau}^\lambda = \left(1, \frac{\partial z(t,x,\lambda)}{\partial x}\right)^T$ be the unit tangent vector to $\mathcal{L}_t^\lambda$, scaled by the element of arclength. It can then be shown that (12) is equivalent to defnining

$$K^\lambda = \mathbf{u}\left(t,x,\lambda\right) \cdot \boldsymbol{\tau}^\lambda\Big|_{\mathcal{L}_t^\lambda}.$$

At the free surface $z = z\left(t,x,1\right) = h(t,x)$, the pressure vanishes, and we obtain the simpler conservation law

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial x}\left(uK + gh - \frac{1}{2}(u^2 + w^2)\right) = 0. \tag{14}$$

This is an exact conservation law representing the evolution of the tangent velocity K along the free surface. While this conservation law can be related to the densities found in [6] in the case of potential flow (since then $K = \partial_x \phi(t,x,\eta(t,x))$, it is clear from the preceding discussion that (14) holds even in the case of rotational flows. In fact, the equation for K corresponds to the equation (9) in [14] where a new Hamiltonian formulation of the water waves equations in the presence of vorticity was proposed. To the best of our knowledge, the general equation (13) valid for each material curve $\mathcal{L}_t^\lambda$ is new, and its derivation has a clear geometrical meaning. In the case where the conservation law is evaluated at the free surface, a geometric interpretation was also provided in [3].

2.2 2D balance law in the Green-Naghdi approximation

In the case of potential motion the exact balance law (14) can be related to the fourth conservation law (6) satisfied by solutions to the Green-Naghdi system (1)-(2) by truncating the resulting set of dimensionless equations at second order in β. Indeed, we recall the scaling which is used in the derivation of the system (1)-(2). Define the non-dimensional variables

$$\tilde{x} = \frac{x}{l}, \quad \tilde{z} = \frac{z}{h_0}, \quad \tilde{t} = \frac{c_0}{l}t, \quad \tilde{h} = \frac{h}{h_0}, \quad \tilde{u} = \frac{u}{c_0}, \quad \tilde{w} = \frac{w}{\sqrt{\beta}c_0}, \quad \tilde{P} = \frac{P}{\rho g h_0}. \tag{15}$$

Using this scaling, the quantity $K = u + wh_x|_{z=h(t,x)}$ is given in non-dimensional variables as

$$\tilde{K}\left(\tilde{t}, \tilde{x}\right) = \tilde{u} + \beta \tilde{w} \tilde{h}_{\tilde{x}},$$

and the conservation law (14) turns into

$$\frac{\partial \tilde{K}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}\Big(\tilde{K}\tilde{u} + \tilde{h} - \frac{1}{2}(\tilde{u}^2 + \beta \tilde{w}^2)\Big) = 0.$$

In the case of potential flows, the expressions of the horizontal and vertical velocity are as follows (see, for example, [5]):

$$\tilde{u}\left(x, z, t\right) = \tilde{\bar{u}} + \beta\Big(\frac{\tilde{h}^2}{6} - \frac{\tilde{z}^2}{2}\Big)\tilde{\bar{u}}_{xx} + \mathcal{O}(\beta^2),$$

$$\tilde{w}(\tilde{x}, \tilde{z}, \tilde{t}) = -\tilde{z}\tilde{\bar{u}}_x + \mathcal{O}(\beta).$$

Substituting the expressions for $\tilde{u}$ and $\tilde{w}$ evaluated at the free surface into the formula for $\tilde{K}\left(\tilde{t}, \tilde{x}\right)$ yields

$$\frac{\partial}{\partial \tilde{t}}\Big(\tilde{\bar{u}} - \frac{\beta}{3}\tilde{h}^2\tilde{\bar{u}}_{\tilde{x}\tilde{x}} - \beta\tilde{h}\tilde{h}_{\tilde{x}}\tilde{\bar{u}}_{\tilde{x}}\Big) + \frac{\partial}{\partial \tilde{x}}\left(\frac{1}{2}\tilde{\bar{u}}^2 - \frac{\beta}{3}\tilde{h}^2\tilde{\bar{u}}\tilde{\bar{u}}_{\tilde{x}\tilde{x}} - \beta\tilde{h}\tilde{h}_{\tilde{x}}\tilde{\bar{u}}\tilde{\bar{u}}_{\tilde{x}} + \tilde{h} - \frac{\beta}{2}\tilde{h}^2\tilde{\bar{u}}_{\tilde{x}}^2\right) = \mathcal{O}(\beta^2).$$

Using similar reasoning as in [1, 2], we define the non-dimensional quantities $\tilde{\mathcal{K}}$ and $\tilde{q}_{\mathcal{K}}$ by

$$\tilde{\mathcal{K}} = \left(\tilde{\bar{u}} - \frac{\beta}{3}\tilde{h}^2\tilde{\bar{u}}_{\tilde{x}\tilde{x}} - \beta\tilde{h}\tilde{h}_{\tilde{x}}\tilde{\bar{u}}_{\tilde{x}}\right)$$

and

$$\tilde{q}_{\mathcal{K}} = \left((\frac{1}{2}\tilde{\bar{u}}^2 - \tfrac{\beta}{3}\tilde{h}^2\tilde{\bar{u}}\tilde{\bar{u}}_{\tilde{x}\tilde{x}} - \beta\tilde{h}\tilde{h}_{\tilde{x}}\tilde{\bar{u}}\tilde{\bar{u}}_{\tilde{x}}) + \tilde{h} - \frac{\beta}{2}\tilde{h}^2\tilde{\bar{u}}_{\tilde{x}}^2\right).$$

Then the balance is

$$\frac{\partial \tilde{\mathcal{K}}}{\partial \tilde{t}} + \frac{\partial \tilde{q}_{\mathcal{K}}}{\partial \tilde{x}} = \mathcal{O}(\beta^2).$$

Using the natural scalings $\mathcal{K} = c_0\tilde{\mathcal{K}}$ and $q_{\mathcal{K}} = c_0^2\tilde{q}_{\mathcal{K}}$, the dimensional forms of these quantities are given by

$$\mathcal{K} = \bar{u} - \frac{1}{3}h^2\bar{u}_{xx} - hh_x\bar{u}_x,$$

$$q_{\mathcal{K}} = gh + \frac{1}{2}\bar{u}^2 - \frac{1}{3}h^2\bar{u}\bar{u}_{xx} - hh_x\bar{u}\bar{u}_x - \frac{1}{2}h^2\bar{u}_x^2,$$

respectively. While the preceding analysis shows that the conservation law

$$\frac{\partial \mathcal{K}}{\partial t} + \frac{\partial q_{\mathcal{K}}}{\partial x} = 0, \tag{16}$$

holds approximately to second order in β, it can be seen immediately that the density $\mathcal{K}$ and the flux $q_{\mathcal{K}}$ are the same as the respective quantities in (6), so that (16) is an exact identity for solutions of the Green-Naghdi system (1)-(2). The asymptotic expansion for K can also be found in (5.17) in [29].

It is interesting to see what happens in the shallow-water approximation where the balance

$$\frac{\partial \tilde{\mathcal{K}}}{\partial \tilde{t}} + \frac{\partial \tilde{q}_{\mathcal{K}}}{\partial \tilde{x}} = \mathcal{O}(\beta)$$

only holds to the order of β. In this case, we obtain

$$\mathcal{K} = \bar{u}$$

for the density and

$$q_{\mathcal{K}} = gh + \frac{1}{2}\bar{u}^2$$

for the flux. Thus in this case the equation $\frac{\partial \mathcal{K}}{\partial t} + \frac{\partial q_{\mathcal{K}}}{\partial x} = 0$ also holds exactly, and in fact it is just the second equation in the shallow-water system. It can be seen clearly from this form that this is a balance law relating horizontal velocity energy density per unit mass. In the shallow-water theory, the horizontal velocity is taken to be uniform throughout the fluid column, so that the simple form for $\mathcal{K}$ and $q_{\mathcal{K}}$ is achieved. However, the quadratic dependence of the horizontal velocity gives a more refined representation which may also be used advantageously for kinematic studies of the fluid motion, such as in [7, 10, 27].

In the derivation of the Green-Naghdi equations (3)-(4), such as summarized in the appendix, no hypothesis about the flow potentiality was used, except that it needs to be assumed that the vorticity be small. Since the conservation laws (4), (5) and (6) are exact consequences of the Green-Nagdhi system, these formulas also hold in the case of rotational flow. However, in order to show the asymptotic equivalence of (14) and (6), the horizontal velocity needed to be assumed to vary quadratically with the depth, and this in turn depends on the flow being irrotational.

To obtain an asymptotic link between (14) and (6) in the general case remains an open problem. Indeed, the vorticity distribution in the fluid layer will certainly influence the average velocity and the surface velocity. In this case, the conservation law (6) might correspond to the equation (13) with some λ depending on the vorticity distribution.

3 Kinematic balance law in three dimensions

3.1 3D balance law for perfect fluids

The Euler equation (8)-(9) are written in general form, and may also be interpreted to hold in three dimensions, for a velocity field $\mathbf{u} = (u, v, w)^T$, a pressure $p(t, x, y, z)$ and body forcing $\mathbf{g} = (0, 0, -g)^T$. In this case, the kinematic free surface boundary condition is given by $\eta_t + u\eta_x + v\eta_y = w$ at $z = \eta(t, x, y)$. Note that for the purpose of this section, we define the horizontal spatial coordinates, velocity vector field, gradient and material derivative by

$$\mathbf{x}_H = (x, y)^T, \quad \mathbf{u}_H = (u, v)^T, \quad \nabla_H = \Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big)^T, \quad \text{and} \quad \frac{D_H}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_H \cdot \nabla_H,$$

respectively.

An analogous vector law for the evolution of the tangent velocity can also be derived in three-dimensional case. Let Σ_t be defined by $z = h(t, \mathbf{x}_H)$, be the free surface, which is assumed can be projected onto the horizontal plane (Oxy). As before, we will introduce a special Lagrangian coordinate $0 \le \lambda \le 1$ corresponding to the foliation of the fluid domain by the material surfaces Σ_t^λ. The free surface Σ_t will correspond to the value $\lambda = 1$. Such a foliation can be achieved by solving the Cauchy problem for the equation

$$\frac{\partial \Phi}{\partial t} + \mathbf{u}_H \cdot \nabla_H \Phi = w,$$

with the initial condition

$$\Phi|_{t=0} = \lambda h_0 + \lambda \eta_0(\mathbf{x}_H),$$

and making then the change of variables $(\mathbf{x}_H, z) \to (\mathbf{x}_H, \lambda)$, $z = \Phi(t, \mathbf{x}_H, \lambda)$.

Then we introduce the Lagrangian coordinates s^1, s^2 which may be taken as the (x, y) coordinates of a particle at time $t = 0$. Each surface Σ_t^λ is thus parametrized by the same parameters s^1, s^2. The change of variables is assumed to be invertible: $s^i = s^i(t, \mathbf{x}_H, \lambda)$, for $i = 1, 2$. The Lagrangian coordinates s^i are conserved along the particle trajectories in the Eulerian coordinates. This implies that the transport equation for the Lagrangian coordinates s^i reduces to

$$\Big(\frac{\partial}{\partial t} + \mathbf{u}_H \cdot \nabla_H\Big) s^i = 0, \qquad i = 1, 2, \tag{17}$$

in the coordinates $(\mathbf{x}_H, \lambda)$. Let

$$\mathbf{e}^i = \nabla_H s^i, \qquad i = 1, 2,$$

be a local *cobasis* at each surface Σ_t^λ (we call it *cobasis* because $\mathbf{e}^i$ are gradients). Taking the horizontal gradient ∇_H of (17) yields

$$\mathbf{e}_t^i + \nabla_H(\mathbf{e}^i \cdot \mathbf{u}_H) = 0. \tag{18}$$

Since the matrix $\frac{\partial \mathbf{e}^i}{\partial \mathbf{x}_H}$ is symmetric, (18) can be written in the form

$$\frac{D_H}{Dt}\mathbf{e}_t^i = \mathbf{e}_t^i + \left(\frac{\partial \mathbf{e}^i}{\partial \mathbf{x}_H}\right)\mathbf{u}_H = -\left(\frac{\partial \mathbf{u}_H}{\partial \mathbf{x}_H}\right)^T \mathbf{e}^i. \tag{19}$$

In analogy with the two-dimensional case, one may take the drift along each s^i- curve containing in the contact surface Σ_t^λ to obtain conservation laws analogous to (7) in the form

$$\frac{\partial\big(\mathbf{u} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial s^i}\big)}{\partial t} + \frac{\partial}{\partial s^i}\left(\frac{p}{\rho} + gz - \frac{|\mathbf{u}|^2}{2}\right) = 0. \tag{20}$$

where $\boldsymbol{\varphi}(t, s^1, s^2, \lambda)$ is the motion of particles along the surfaces Σ_t^λ. Defining

$$K_i^\lambda = \mathbf{u}(t, s^1, s^2, \lambda) \cdot \frac{\partial \boldsymbol{\varphi}(t, s^1, s^2, \lambda)}{\partial s^i}$$

in analogy with the two-dimensional case, we can define the tangent component of the velocity vector at Σ_t^λ as

$$\mathbf{K}^\lambda = K_1^\lambda \mathbf{e}^1 + K_2^\lambda \mathbf{e}^2.$$

Taking the partial derivative of $\mathbf{K}^\lambda(t, s^1, s^2)$ with respect to time at fixed Lagrangian coordinates s^1, s^2 we obtain

$$\frac{\partial \mathbf{K}^\lambda\left(t, s^1, s^2\right)}{\partial t} = \frac{\partial K_1^\lambda}{\partial t}\mathbf{e}^1 + \frac{\partial K_2^\lambda}{\partial t}\mathbf{e}^2 + K_1^\lambda \frac{\partial \mathbf{e}^1(t, s^1, s^2, \lambda)}{\partial t} + K_2^\lambda \frac{\partial \mathbf{e}^2(t, s^1, s^2, \lambda)}{\partial t}$$

$$= -\frac{\partial}{\partial s^1}\left(\frac{p}{\rho}+gz-\frac{|\mathbf{u}|^2}{2}\right)\mathbf{e}^1 - \frac{\partial}{\partial s^2}\left(\frac{p}{\rho}+gz-\frac{|\mathbf{u}|^2}{2}\right)\mathbf{e}^2 + K_1^\lambda \frac{\partial \mathbf{e}^1(t,s^1,s^2,\lambda)}{\partial t} + K_2^\lambda \frac{\partial \mathbf{e}^2(t,s^1,s^2,\lambda)}{\partial t}.$$

Converting to Eulerian coordinates, we obtain

$$\frac{D_H}{Dt}\mathbf{K}^\lambda(t,\mathbf{x}_H) = -\nabla_H\Big(\frac{p}{\rho}+gz-\frac{|\mathbf{u}|^2}{2}\Big) + K_1^\lambda \frac{D_H}{Dt}\mathbf{e}^1(t,\mathbf{x}_H,\lambda) + K_2^\lambda \frac{D_H}{Dt}\mathbf{e}^2(t,\mathbf{x}_H,\lambda).$$

Using (19), we may write the last relation as

$$\frac{D_H}{Dt}\mathbf{K}^\lambda(t,\mathbf{x}_H) = -\nabla_H\left(\frac{p}{\rho}+gz-\frac{|\mathbf{u}|^2}{2}\right) - \left(\frac{\partial \mathbf{u}_H}{\partial \mathbf{x}_H}\right)^T \mathbf{K}^\lambda.$$

This equation may also be written as

$$\frac{\partial \mathbf{K}^\lambda(t,\mathbf{x}_H)}{\partial t} + \left[\frac{\partial \mathbf{K}^\lambda}{\partial \mathbf{x}_H} - \left(\frac{\partial \mathbf{K}^\lambda}{\partial \mathbf{x}_H}\right)^T\right]\mathbf{u}_H + \left(\frac{\partial \mathbf{K}^\lambda}{\partial \mathbf{x}_H}\right)^T \mathbf{u}_H + \left(\frac{\partial \mathbf{u}_H}{\partial \mathbf{x}_H}\right)^T \mathbf{K}^\lambda + \nabla_H\left(\frac{p}{\rho}+gz-\frac{|\mathbf{u}|^2}{2}\right) = 0,$$

or equivalently as

$$\frac{\partial \mathbf{K}^\lambda(t,\mathbf{x}_H)}{\partial t} + \mathrm{curl}(\mathbf{K}^\lambda)\times \mathbf{u}_H + \nabla_H\left\{\mathbf{K}^\lambda\cdot\mathbf{u}_H + \frac{p}{\rho}+gz-\frac{|\mathbf{u}|^2}{2}\right\} = 0. \qquad (21)$$

Equation (21) holds on the material surface Σ_t^λ, and is the equivalent of (13) in the three-dimensional setting.

At the free surface $z = h(t,\mathbf{x}_H)$ equation (21) can be writen in a simpler form. Indeed, since at the free surface $p = 0$ and

$$h_t + \mathbf{u}_H\cdot\nabla_H h = w,$$

where $\mathbf{u}_H$ is evaluated at the free surface and w is the vertical velocity evaluated at the free surface, (21) may be rewritten in the form

$$\frac{\partial \mathbf{K}}{\partial t} + \mathrm{curl}(\mathbf{K})\times \mathbf{u}_H + \nabla_H\left\{\mathbf{K}\cdot\mathbf{u}_H + gh - \frac{1}{2}\left(|\mathbf{u}_H|^2 + \left(\tfrac{D_H h}{Dt}\right)^2\right)\right\} = 0, \qquad (22)$$

which is analogous to (14). Since $\mathbf{K}$ is a vector in the $(x,y)-$ plane, the vector $\mathrm{curl}(\mathbf{K})$ has only one non-zero component and is orthogonal to the $(x,y)-$ plane. Equation (22) is exactly the equation proposed in [14] (see equation (9) in that paper written by using different notations).

Two observations are in order. First of all, equation (22) holds in the case of non-zero vorticity as well as the case of potential flow. In the latter case, we have $\mathrm{curl}(\mathbf{K})\times\mathbf{u}_H = 0$, since

$$\mathrm{curl}(\mathbf{K})\times \mathbf{u}_H = \Big(\mathrm{curl}(\mathbf{u})|_{z=h(t,\mathbf{x}_H)}\cdot \mathbf{N}\Big)\,\mathbf{u}_H^\perp, \quad \mathbf{N} = (-\nabla_H h, 1)^T, \quad \mathbf{u}_H^\perp = (-v,u)^T,$$

so that (22) is a proper conservation law in the case of potential motion.

Secondly, it will be convenient to express $\mathbf{K}$ in terms of the velocity field in Eulerian coordinates. For this purpose, note the following computation:

$$\mathbf{K} = \left(\mathbf{u}_H\cdot\frac{\partial \mathbf{x}_H}{\partial s^1} + w\frac{\partial h}{\partial s^1}\right)\nabla_H s^1 + \left(\mathbf{u}_H\cdot\frac{\partial \mathbf{x}_H}{\partial s^2} + w\frac{\partial h}{\partial s^2}\right)\nabla_H s^2$$

$$= \left(\nabla_H s^1 \otimes \frac{\partial \mathbf{x}_H}{\partial s^1} + \nabla_H s^2 \otimes \frac{\partial \mathbf{x}_H}{\partial s^2}\right)\mathbf{u}_H + w\left(\frac{\partial h}{\partial s^1}\nabla_H s^1 + \frac{\partial h}{\partial s^2}\nabla_H s^2\right).$$

$$= \mathbf{u}_H + w\nabla_H h.$$

3.2 3D balance law in the Green-Naghdi approximation

Next, we consider the approximation of (22) in the Green-Naghdi regime. Recall that the Green-Naghdi equations for three-dimensional flows are written in terms of the depth-averaged horizontal velocity $\bar{\mathbf{u}}$ and the total flow depth h as

$$\eta_t + \nabla_H \cdot (h\bar{\mathbf{u}}) = 0, \tag{23}$$

$$\left\{\bar{\mathbf{u}}_t + \mathcal{T}(h, \bar{\mathbf{u}}_t)\right\} + \bar{\mathbf{u}} \cdot \nabla_H \bar{\mathbf{u}} + \nabla_H \eta + Q(h, \bar{\mathbf{u}}) = 0, \tag{24}$$

where

$$\mathcal{T}(h, \bar{\mathbf{u}}_t) = -\frac{1}{3h}\nabla_H(h^3 \nabla_H \cdot \bar{\mathbf{u}}_t),$$

$$Q(h, \bar{\mathbf{u}}) = \frac{-1}{3h}\nabla_H\Big[h^3\Big(\bar{\mathbf{u}} \cdot \nabla_H(\nabla_H \cdot \bar{\mathbf{u}}) - (\nabla_H \cdot \bar{\mathbf{u}})^2\Big)\Big].$$

In order to obtain an unbiased representation, we write the non-dimensionalized version of (22) in terms of the principal unknown variables of the Green-Naghdi equation. Thus in non-dimensional form, (22) becomes

$$\frac{\partial}{\partial \tilde{t}}\big(\tilde{\mathbf{u}}_H + \beta \tilde{w} \tilde{\nabla}_H \tilde{h}\big) + \widetilde{\operatorname{curl}}\big(\tilde{\mathbf{u}}_H + \beta \tilde{w} \tilde{\nabla}_H \tilde{h}\big) \times \tilde{\mathbf{u}}_H \\ + \tilde{\nabla}_H \left\{\tilde{\mathbf{u}}_H + \beta(\tilde{w}\tilde{\nabla}_H \tilde{h}) \cdot \tilde{\mathbf{u}}_H + \tilde{h} - \frac{1}{2}\Big(|\tilde{\mathbf{u}}_H|^2 + \beta \tilde{w}^2\Big)\right\} = 0. \tag{25}$$

Note that since the velocity fields are now approximated, the term $\operatorname{curl}(\mathbf{K}) \times \mathbf{u}_H$ may not evaluate to zero even for irrotational flow, and therefore needs to included.

Similar to the two-dimensional case, in the case of potential flow, the non-dimensional horizontal velocity field $\tilde{\mathbf{u}}_H$, and the vertical velocity component $\tilde{w}$ are given in the Green-Naghdi approximation at any location $(\tilde{\mathbf{x}}_H, \tilde{z})$ in the fluid by the following expressions in terms of the depth averaged velocity:

$$\tilde{\mathbf{u}}_H(\tilde{\mathbf{x}}, \tilde{z}, \tilde{t}) = \tilde{\bar{\mathbf{u}}} + \beta\Big(\frac{\tilde{h}^2}{6} - \frac{\tilde{z}^2}{2}\Big)\tilde{\Delta}_H \tilde{\bar{\mathbf{u}}} + \mathcal{O}(\beta^2), \tag{26}$$

$$\tilde{w}(\tilde{\mathbf{x}}_H, \tilde{z}, \tilde{t}) = -\tilde{z}\left(\nabla_H \cdot \tilde{\bar{\mathbf{u}}}\right) + \mathcal{O}(\beta). \tag{27}$$

Moreover, in the Appendix, the following important relation is established:

$$\tilde{\Delta}_H \tilde{\bar{\mathbf{u}}} = \tilde{\nabla}_H(\tilde{\nabla}_H \cdot \tilde{\bar{\mathbf{u}}}) + \mathcal{O}(\beta).$$

This relation holds as long as irrotational solutions of Euler's equations are considered. We now look at the three terms in equation (25) individually. The first term can be written as

$$\frac{\partial}{\partial \tilde{t}}\big(\tilde{\mathbf{u}}_H + \beta \tilde{w}\tilde{\nabla}_H \tilde{h}\big) = \frac{\partial}{\partial \tilde{t}}\Big(\tilde{\bar{\mathbf{u}}} - \frac{\beta}{3}\tilde{h}^2 \tilde{\Delta}\tilde{\bar{\mathbf{u}}} - \beta \tilde{h}(\tilde{\nabla}_H \cdot \tilde{\bar{\mathbf{u}}})\nabla_H \tilde{h}\Big) + \mathcal{O}(\beta^2),$$

which suggests defining

$$\tilde{\mathbb{K}} = \tilde{\bar{\mathbf{u}}} - \frac{\beta}{3\tilde{h}}\tilde{\nabla}_H\left(\tilde{h}^3 \tilde{\operatorname{div}}_H \tilde{\bar{\mathbf{u}}}\right), \tag{28}$$

With this definition, the second term can be written as

$$\begin{aligned}
\widetilde{\operatorname{curl}}\big(\tilde{\mathbf{u}}_H + \beta\tilde{\nabla}_H\tilde{h}\big) \times \tilde{\mathbf{u}}_H &= \begin{bmatrix} \frac{\partial}{\partial\tilde{y}}(\tilde{u} + \beta\tilde{w}\tilde{h}_{\tilde{x}})\tilde{v} - \frac{\partial}{\partial\tilde{x}}(\tilde{v} + \beta\tilde{w}\tilde{h}_{\tilde{y}})\tilde{v} \\ \frac{\partial}{\partial\tilde{x}}(\tilde{v} + \beta\tilde{w}\tilde{h}_{\tilde{y}})\tilde{u} - \frac{\partial}{\partial\tilde{y}}(\tilde{u} + \beta\tilde{w}\tilde{h}_{\tilde{x}})\tilde{u} \end{bmatrix} \\
&= \begin{bmatrix} \Big(\frac{\partial}{\partial\tilde{y}}\tilde{\mathbb{K}}_1 - \frac{\partial}{\partial\tilde{x}}\tilde{\mathbb{K}}_2\Big)\tilde{\bar{u}} \\ \Big(\frac{\partial}{\partial\tilde{x}}\tilde{\mathbb{K}}_2 - \frac{\partial}{\partial\tilde{y}}\tilde{\mathbb{K}}_1\Big)\tilde{\bar{v}} \end{bmatrix} + \beta\frac{\tilde{h}^2}{3} \begin{bmatrix} \Big(\frac{\partial\tilde{\bar{v}}}{\partial\tilde{x}} - \frac{\partial\tilde{\bar{u}}}{\partial\tilde{y}}\Big)\tilde{\Delta}_H\tilde{\bar{v}} \\ \Big(\frac{\partial\tilde{\bar{u}}}{\partial\tilde{y}} - \frac{\partial\tilde{\bar{v}}}{\partial\tilde{x}}\Big)\tilde{\Delta}_H\tilde{\bar{u}} \end{bmatrix} + \mathcal{O}(\beta^2) \\
&= \begin{bmatrix} \Big(\frac{\partial}{\partial\tilde{y}}\tilde{\mathbb{K}}_1 - \frac{\partial}{\partial\tilde{x}}\tilde{\mathbb{K}}_2\Big)\tilde{\bar{u}} \\ \Big(\frac{\partial}{\partial\tilde{x}}\tilde{\mathbb{K}}_2 - \frac{\partial}{\partial\tilde{y}}\tilde{\mathbb{K}}_1\Big)\tilde{\bar{v}} \end{bmatrix} + \mathcal{O}(\beta^2),
\end{aligned}$$

where we have used the fact (shown in the appendix) that $\frac{\partial\tilde{\bar{u}}}{\partial\tilde{y}} - \frac{\partial\tilde{\bar{v}}}{\partial\tilde{x}} = \mathcal{O}(\beta)$. Finally, the last term in (25) can be written as

$$\begin{aligned}
\tilde{\nabla}_H\bigg\{ \Big(\tilde{\bar{\mathbf{u}}} - \tfrac{\beta}{3}\tilde{h}^2\Delta\tilde{\bar{\mathbf{u}}} - \beta\tilde{h}(\nabla_H\cdot\tilde{\bar{\mathbf{u}}})\nabla_H\tilde{h} + \mathcal{O}(\beta^2)\Big)\cdot\Big(\tilde{\bar{\mathbf{u}}} - \beta\tfrac{\tilde{h}^2}{3}\tilde{\Delta}_H\tilde{\bar{\mathbf{u}}} + \mathcal{O}(\beta^2)\Big) + \tilde{h} \\
- \frac{1}{2}\Big(|\tilde{\bar{\mathbf{u}}}|^2 - 2\beta\tfrac{\tilde{h}^2}{3}\tilde{\bar{\mathbf{u}}}\cdot\tilde{\Delta}\tilde{\bar{\mathbf{u}}} + \beta|\tilde{h}(\tilde{\nabla}_H\cdot\tilde{\bar{\mathbf{u}}})|^2 + \mathcal{O}(\beta^2)\Big)\bigg\} \\
= \tilde{\nabla}_H\bigg\{\tilde{\mathbb{K}}\cdot\tilde{\bar{\mathbf{u}}} + \tilde{h} - \frac{1}{2}\Big(|\tilde{\bar{\mathbf{u}}}|^2 + \beta|\tilde{h}(\tilde{\nabla}_H\cdot\tilde{\bar{\mathbf{u}}})|^2\Big)\bigg\} + \mathcal{O}(\beta^2).
\end{aligned}$$

Thus with the definition of $\mathbb{K}$ as in (28), the relation (25) can be written as

$$\frac{\partial}{\partial\tilde{t}}\tilde{\mathbb{K}} + \widetilde{\operatorname{curl}}(\tilde{\mathbb{K}})\times\tilde{\bar{\mathbf{u}}} + \tilde{\nabla}_H\left\{\tilde{\mathbb{K}}\cdot\tilde{\bar{\mathbf{u}}} + \tilde{h} - \frac{1}{2}\beta\tilde{h}^2|\tilde{\nabla}_H\cdot\tilde{\bar{\mathbf{u}}}|^2\right\} = \mathcal{O}(\beta^2). \tag{29}$$

This equation is the approximate form of (21) which is valid to the same order of approximation as the Green-Naghdi system. However, as in the two-dimensional case, it appears that the equation is actually an exact consequence of the Green-Naghdi equations, and the $\mathcal{O}(\beta^2)$ term in (29) may be replaced by zero. Then, using the equation (23), the balance law (29) may be written in dimensional variables as

$$\frac{\partial\mathbb{K}}{\partial t} + \operatorname{curl}(\mathbb{K})\times\bar{\mathbf{u}} + \nabla_H\Big\{\mathbb{K}\cdot\bar{\mathbf{u}} + gh - \frac{1}{2}\Big(|\bar{\mathbf{u}}|^2 + \big(\tfrac{D_H h}{Dt}\big)^2\Big)\Big\} = 0, \tag{30}$$

where $\mathbb{K} = \bar{\mathbf{u}} - \frac{1}{3h}\nabla_H\left(h^3\nabla_H\cdot\bar{\mathbf{u}}\right)$ is the dimensional form of (28), and the material derivative in this context is defined as $\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{\mathbf{u}}\cdot\nabla_H$.

4 Discussion

In both potential and rotational flows, solutions of the water wave problem satisfy the exact balance equation

$$\frac{\partial\big(\mathbf{u}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial s^i}\big)}{\partial t} + \frac{\partial}{\partial s^i}\Big(\frac{p}{\rho} + gz - \frac{|\mathbf{u}|^2}{2}\Big) = 0,$$

where s_i are Lagrangian coordinates, and $\boldsymbol{\varphi}$ denotes an ensemble of particle paths. This equation holds both in two and in three dimensions. In two dimensions, the above relation gives rise to the differential conservation equation (14) involving the tangential velocity along the free surface which we denoted by K. In three dimensions, the somewhat more complicated equation (21) appears.

These conservation equations have been studied in the Green-Naghdi approximation which assumes that surface waves are long when compared to the undisturbed depth of the fluid, and where terms of order higher than β, where $\beta = h_0^2/l^2$ is the long-wave parameter, are disregarded. In the case of potential flow, it was found that these equations give rise to corresponding approximate balance laws which have the same form as the exact conservation equations (6) and (28) of the Green-Naghdi equations in two and three dimensions, respectively. In this respect, we have completed the physical interpretation of the conservation laws connected with the Green-Naghdi equations in the case of potential flow.

To summarize the conservation laws in the three-dimensional setting, note that solutions of the Green-Naghdi equations admit conservation of mass, given by

$$\frac{\partial h}{\partial t} + \nabla_H \cdot (h\bar{\mathbf{u}}) = 0.$$

Conservation of momentum is represented by

$$\frac{\partial}{\partial t}\left\{h\bar{\mathbf{u}}\right\} + \nabla \cdot \left\{h\bar{\mathbf{u}} \otimes \bar{\mathbf{u}} + \left(g\frac{h^2}{2} + \frac{1}{3}h^2\frac{D^2h}{Dt^2}\right)\mathbb{I}\right\} = 0,$$

where $\frac{D^2}{Dt^2} = \left(\frac{\partial}{\partial t} + \bar{\mathbf{u}} \cdot \nabla_H\right)\left(\frac{\partial}{\partial t} + \bar{\mathbf{u}} \cdot \nabla_H\right)$, and $\mathbb{I}$ is the identity matrix. Conservation of energy in the three-dimensional Green-Naghdi system is represented by the following conservation law:

$$\frac{\partial}{\partial t}\left\{h\frac{|\bar{\mathbf{u}}|^2}{2} + E\right\} + \nabla_H \cdot \left\{\bar{\mathbf{u}}\left(h\frac{|\bar{\mathbf{u}}|^2}{2} + E + p\right)\right\} = 0,$$

where

$$E = \frac{gh^2}{2} + \frac{h}{6}\left(\frac{Dh}{Dt}\right)^2.$$

Finally, the equations admit the exact consequence (30):

$$\frac{\partial \mathbb{K}}{\partial t} + \mathrm{curl}(\mathbb{K}) \times \bar{\mathbf{u}} + \nabla_H\left\{\mathbb{K}\cdot\bar{\mathbf{u}} + gh - \frac{1}{2}\left(|\bar{\mathbf{u}}|^2 + \left(\frac{Dh}{Dt}\right)^2\right)\right\} = 0,$$

with $\mathbb{K} = \bar{\mathbf{u}} - \frac{1}{3h}\nabla_H\left(h^3\nabla_H \cdot \bar{\mathbf{u}}\right) = \bar{\mathbf{u}} + \frac{1}{3h}\nabla_H\left(h^2\frac{Dh}{Dt}\right)$. This conservation law is obtained as an approximation of a general exact balance law (22) at the free surface for the full Euler equations in the case of potential motions. Since both (22) and (30) also hold in the the case of rotational flow, a natural question is if the asymptotic equivalence can be extended to this more general case. One possible approach would be to use ideas from [14] where a fully nonlinear long-wave system is derived in the case of non-potential flow. In this approximation, the velocity field below the surface is governed by an additional evolution equation, and this equation could be the basis for finding an approximation to the velocity field in the rotational case. However, this is beyond the scope of the present article.

The importance of the variable $\mathbb{K}$ for numerical purposes is demonstrated in [31, 35]. Analogous equations can also be obtained in the case of multi-layer flows, such as for instance in [4] and [38].

Acknowledgments. The autors thank reviewers for valuables comments and suggestions. SG was partially supported by ANR BoND, France. HK and ZK acknowledge support by the Research Council of Norway.

5 Appendix

5.1 Derivation of the 2D Green-Naghdi equations

The derivation of the Green-Naghdi equations in two spatial dimensions can be found in [5] in the case of potential flows. The derivation in the case of non-potential flows can be found, in particular, in [4] where a variational approach was used based on the Hamilton principle. However, a direct averaging the Euler equations can also be done. Here we will indicate some of the key steps for the vorticity case. In order to understand the approximation of (14) in the Green-Naghdi theory, we recall the scaling which is used in the derivation of the system (1)-(2). Using the non-dimensionalization (15) from Section 2.2, and averaging the continuity equation (9) over the local depth of the fluid leads to the continuity equation (1), where the average horizontal velocity $\tilde{\bar{u}}$ is defined by

$$\tilde{\bar{u}} = \frac{1}{\tilde{h}} \int_0^{\tilde{h}} \tilde{u} d\tilde{z}.$$

Using the boundary conditions, the continuity equation and depth averaged values, the momentum equation

$$\tilde{\bar{u}}_{\tilde{t}} + \tilde{\bar{u}}\tilde{\bar{u}}_{\tilde{x}} + \tilde{\eta}_{\tilde{x}} + \frac{\beta}{\tilde{h}} \frac{\partial}{\partial \tilde{x}} \int_0^{\tilde{h}} \tilde{z}\Gamma(\tilde{x}, \tilde{z}, \tilde{t})\, d\tilde{z} = -\frac{1}{\tilde{h}} \frac{\partial}{\partial \tilde{x}} \int_0^{\tilde{h}} \left(\tilde{u}^2 - \tilde{\bar{u}}^2\right)\, dz$$

appears, where

$$\Gamma(\tilde{x}, \tilde{z}, \tilde{t}) = \tilde{w}_{\tilde{t}} + \tilde{u}\tilde{w}_{\tilde{x}} + \tilde{w}\tilde{w}_{\tilde{z}}$$

is the vertical acceleration. The estimation of the correlation

$$\int_0^{\tilde{h}} \left(\tilde{u}^2 - \tilde{\bar{u}}^2\right)\, dz$$

is now needed. In the two-dimensional case, the vorticity has only one scalar component $\tilde{\omega}$ which satisfies the equation

$$\tilde{\omega}_{\tilde{t}} + \tilde{u}\tilde{\omega}_{\tilde{x}} + \tilde{w}\tilde{\omega}_{\tilde{z}} = 0,$$

where

$$\tilde{\omega} = \tilde{u}_{\tilde{z}} - \beta \tilde{w}_{\tilde{x}}.$$

This implies the following equation for $\tilde{\Omega} = \tilde{u}_{\tilde{z}}$:

$$\tilde{\Omega}_{\tilde{t}} + \tilde{u}\tilde{\Omega}_{\tilde{x}} + \tilde{w}\tilde{\Omega}_{\tilde{z}} = \mathcal{O}(\beta).$$

If, initially, $\tilde{\Omega}\Big|_{t=0} = \mathcal{O}(\beta^s)$, $s > 0$, then for any time

$$\tilde{\Omega} = \mathcal{O}(\beta^{\min(s,1)}).$$

In particular, it implies (see details in [4]) that

$$|\tilde{u} - \tilde{\bar{u}}| = \mathcal{O}(\beta^{\min(s,1)}),$$

$$\int_0^{\tilde{h}} \left(\tilde{u}^2 - \tilde{\bar{u}}^2\right)\, dz = \mathcal{O}(\beta^{2\min(s,1)}).$$

Since the principal term of $\tilde{w}$ is in the form

$$\tilde{w}(\tilde{x},\tilde{z},\tilde{t}) = -\tilde{z}\frac{\partial\tilde{\bar{u}}}{\partial\tilde{x}} + \mathcal{O}(\beta^{\min(s,1)}),$$

and therefore,

$$\Gamma(\tilde{x},\tilde{z},\tilde{t}) = -\tilde{z}\left(\tilde{\bar{u}}_{\tilde{x}\tilde{t}} + \tilde{\bar{u}}\tilde{\bar{u}}_{\tilde{x}\tilde{x}} - \tilde{\bar{u}}_{\tilde{x}}^2\right) + \mathcal{O}(\beta^{\min(s,1)}),$$

we find in the case $s > 1/2$ the equation

$$\tilde{\bar{u}}_{\tilde{t}} + \tilde{\bar{u}}\tilde{\bar{u}}_{\tilde{x}} + \tilde{\eta}_{\tilde{x}} - \frac{\beta}{3\tilde{h}}\frac{\partial}{\partial\tilde{x}}\left(\tilde{h}^3\left(\tilde{\bar{u}}_{\tilde{x}\tilde{t}} + \tilde{\bar{u}}\tilde{\bar{u}}_{\tilde{x}\tilde{x}} - \tilde{\bar{u}}_{\tilde{x}}^2\right)\right) = \mathcal{O}(\beta^{2\min(s,1)})$$

valid up to order β. So, the derivation of the equation for the average velocity does not require the potentiality hypothesis, the vorticity should only to be quite small.

5.2 Derivation of the 3D Green-Naghdi equations

In the three-dimensional case we use the additional scalings

$$\tilde{y} = \frac{y}{l} \quad \text{and} \quad \tilde{v} = \frac{v}{c_0},$$

so that we can write the horizontal velocity in the form $\tilde{\mathbf{u}}_H = \frac{1}{c_0}\mathbf{u}_H$. Averaging over the local depth of the fluid leads to the continuity equation

$$\tilde{\eta}_{\tilde{t}} + \tilde{\nabla}_H \cdot (\tilde{h}\tilde{\bar{\mathbf{u}}}) = 0, \tag{31}$$

where $\tilde{\bar{\mathbf{u}}} = (\tilde{\bar{u}}, \tilde{\bar{v}})$ is the averaged non-dimensional horizontal velocity: $\tilde{\bar{\mathbf{u}}} = \tilde{\bar{\mathbf{u}}} = \frac{1}{\tilde{h}}\int_0^{\tilde{h}} \tilde{\mathbf{u}}_H\, d\tilde{z}$. Using the boundary conditions, the continuity equation and depth averaged values, the momentum equations

$$\tilde{\bar{u}}_{\tilde{t}} + (\tilde{\bar{\mathbf{u}}}\cdot\tilde{\nabla}_H)\tilde{\bar{u}} + \tilde{\eta}_{\tilde{x}} + \frac{\beta}{\tilde{h}}\frac{\partial}{\partial\tilde{x}}\int_0^{\tilde{h}} \tilde{z}\Gamma(\tilde{x},\tilde{y},\tilde{z},\tilde{t})\,d\tilde{z} =$$

$$-\frac{1}{\tilde{h}}\frac{\partial}{\partial\tilde{x}}\int_0^{\tilde{h}}\left(\tilde{u}^2 - \tilde{\bar{u}}^2\right)\,d\tilde{z} - \frac{1}{\tilde{h}}\frac{\partial}{\partial\tilde{y}}\int_0^{\tilde{h}}\left((\tilde{u}\tilde{v})^2 - \tilde{\bar{u}}^2\right)\,d\tilde{z},$$

and

$$\tilde{\bar{v}}_{\tilde{t}} + (\tilde{\bar{\mathbf{u}}}\cdot\tilde{\nabla}_H)\tilde{\bar{v}} + \tilde{\eta}_{\tilde{y}} + \frac{\beta}{h}\frac{\partial}{\partial\tilde{x}}\int_0^{\tilde{h}} \tilde{z}\Gamma(\tilde{x},\tilde{y},\tilde{z},\tilde{t}))\,d\tilde{z} =$$

$$-\frac{1}{\tilde{h}}\frac{\partial}{\partial\tilde{x}}\int_0^{\tilde{h}}\left(\tilde{v}^2 - \tilde{\bar{v}}^2\right)\,d\tilde{z} - \frac{1}{\tilde{h}}\frac{\partial}{\partial\tilde{y}}\int_0^{\tilde{h}}\left((\tilde{u}\tilde{v})^2 - (\tilde{\bar{v}})^2\right)\,d\tilde{z},$$

appear, where

$$\Gamma(\tilde{x},\tilde{y},\tilde{z},\tilde{t}) = \tilde{w}_{\tilde{t}} + \tilde{u}\tilde{w}_{\tilde{x}} + \tilde{v}\tilde{w}_{\tilde{y}} + \tilde{w}\tilde{w}_{\tilde{z}}.$$

As in 2D case, no need to suppose that the flow is potential. The only hypothesis is that the dimensionless vertical gradient of the horizontal velocity $\tilde{\mathbf{u}}_H$ is initially small (see details in [4]) :

$$\left.\frac{\partial\tilde{\mathbf{u}}_H}{\partial\tilde{z}}\right|_{t=0} = \mathcal{O}(\beta^s), \quad s > 1/2.$$

Then (see [4]) :

$$\left|\tilde{\mathbf{u}}_H - \tilde{\bar{\mathbf{u}}}_H\right| = \mathcal{O}(\beta^{\min(s,1)}),$$

$$\int_0^{\tilde{h}} \left(\tilde{u}^2 - \tilde{\bar{u}}^2\right)\, dz = \mathcal{O}(\beta^{2\min(s,1)}), \quad \int_0^{\tilde{h}} \left((\tilde{u}\tilde{v})^2 - \tilde{\bar{u}}^2\right) = \mathcal{O}(\beta^{2\min(s,1)}), \quad \int_0^{\tilde{h}} \left(\tilde{v}^2 - \tilde{\bar{v}}^2\right)\, d\tilde{z} = \mathcal{O}(\beta^{2\min(s,1)}).$$

and that the vertical velocity becomes

$$\tilde{w}(\tilde{x}, \tilde{z}, \tilde{t}) = -\tilde{z}\tilde{\nabla}_H \cdot \tilde{\bar{\mathbf{u}}} + \mathcal{O}(\beta^{\min(s,1)}).$$

Therefore,

$$\Gamma(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = -\tilde{z}\left[\frac{D}{Dt}\left(\tilde{\nabla}_H \cdot \tilde{\bar{\mathbf{u}}}\right) - \left(\tilde{\nabla}_H \cdot \tilde{\bar{\mathbf{u}}}\right)^2\right] + \mathcal{O}(\beta^{\min(s,1)}), \qquad \frac{D}{D\tilde{t}} = \frac{\partial}{\partial \tilde{t}} + \tilde{\bar{\mathbf{u}}} \cdot \tilde{\nabla}_H,$$

and we find the equation

$$\left\{\tilde{\bar{\mathbf{u}}}_{\tilde{t}} + \beta\tilde{\mathcal{T}}(\tilde{h}, \tilde{\bar{\mathbf{u}}}_{\tilde{t}})\right\} + \tilde{\bar{\mathbf{u}}} \cdot \tilde{\nabla}_H \tilde{\bar{\mathbf{u}}} + \tilde{\nabla}_H \tilde{\eta} + \beta Q(\tilde{h}, \tilde{\bar{\mathbf{u}}}) = \mathcal{O}(\beta^{2\min(s,1)}), \tag{32}$$

where

$$\tilde{\mathcal{T}}(h, \mathbf{V}) = -\frac{1}{3h}\tilde{\nabla}_H(h^3\tilde{\nabla}_H \cdot \mathbf{V}),$$

$$\tilde{Q}(h, \mathbf{V}) = -\frac{1}{3h}\tilde{\nabla}_H\Big[h^3\Big(\mathbf{V} \cdot \tilde{\nabla}_H(\tilde{\nabla}_H \cdot \mathbf{V}) - (\tilde{\nabla}_H \cdot \mathbf{V})^2\Big)\Big].$$

5.3 Alternative representation of the horizontal velocity

An alternative representation of the horizontal velocity field can be obtained by using the relation

$$\tilde{\Delta}_H \tilde{\bar{\mathbf{u}}} = \tilde{\nabla}_H \left(\nabla_H \cdot \tilde{\bar{\mathbf{u}}}\right) + \mathcal{O}(\beta),$$

which holds for potential flows. Indeed, note that

$$\begin{aligned}
\tilde{\nabla}_H \left(\tilde{\nabla}_H \cdot \tilde{\bar{\mathbf{u}}}\right) &= \left(\frac{\partial}{\partial \tilde{x}}\left(\frac{\partial \tilde{\bar{u}}}{\partial \tilde{x}} + \frac{\partial \tilde{\bar{v}}}{\partial \tilde{y}}\right), \frac{\partial}{\partial \tilde{y}}\left(\frac{\partial \tilde{\bar{u}}}{\partial \tilde{x}} + \frac{\partial \tilde{\bar{v}}}{\partial \tilde{y}}\right)\right) \\
&= \left(\frac{\partial^2 \tilde{\bar{u}}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{\bar{u}}}{\partial \tilde{y}^2} - \frac{\partial}{\partial \tilde{y}}\left(\frac{\partial \tilde{\bar{u}}}{\partial \tilde{y}} - \frac{\partial \tilde{\bar{v}}}{\partial \tilde{x}}\right), \frac{\partial^2 \tilde{\bar{v}}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{\bar{v}}}{\partial \tilde{y}^2} + \frac{\partial}{\partial \tilde{x}}\left(\frac{\partial \tilde{\bar{u}}}{\partial \tilde{y}} - \frac{\partial \tilde{\bar{v}}}{\partial \tilde{x}}\right)\right) \\
&= \tilde{\Delta}_H \tilde{\bar{\mathbf{u}}} + \left(\frac{\partial \tilde{\bar{\omega}}}{\partial y}, -\frac{\partial \tilde{\bar{\omega}}}{\partial x}\right),
\end{aligned}$$

where we have introduced the the vorticity of mean horizontal velocity:

$$\tilde{\bar{\omega}} = \frac{\partial \tilde{\bar{v}}}{\partial \tilde{x}} - \frac{\partial \tilde{\bar{u}}}{\partial \tilde{y}}.$$

Let us estimate $\tilde{\bar{\omega}}$ under the condition that the instantaneous flow is irrotational: $v_x - u_y = 0$. We have

$$\begin{aligned}
\tilde{\bar{\omega}} &= \frac{\partial}{\partial \tilde{x}}\left(\frac{1}{\tilde{h}}\int_0^{\tilde{h}} \tilde{v} d\tilde{z}\right) - \frac{\partial}{\partial \tilde{y}}\left(\frac{1}{\tilde{h}}\int_0^{\tilde{h}} \tilde{u} d\tilde{z}\right) \\
&= \frac{\tilde{h}_{\tilde{x}}}{\tilde{h}}\tilde{\bar{v}} - \frac{\tilde{h}_{\tilde{y}}}{\tilde{h}}\tilde{\bar{u}} + \frac{1}{\tilde{h}}\int_0^{\tilde{h}} \left(\tilde{v}_{\tilde{x}} - \tilde{u}_{\tilde{y}}\right) d\tilde{z} + \frac{\tilde{h}_{\tilde{x}}}{\tilde{h}}\tilde{v}\left(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{h}(\tilde{t}, \tilde{x}, \tilde{y})\right) - \frac{\tilde{h}_{\tilde{y}}}{\tilde{h}}\tilde{u}\left(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{h}(\tilde{t}, \tilde{x}, \tilde{y})\right) \\
&= -\frac{\tilde{h}_{\tilde{x}}}{\tilde{h}}\left(\tilde{\bar{v}} - \tilde{v}\left(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{h}(\tilde{t}, \tilde{x}, \tilde{y})\right)\right) + \frac{\tilde{h}_{\tilde{y}}}{\tilde{h}}\left(\tilde{\bar{u}} - \tilde{u}\left(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{h}(\tilde{t}, \tilde{x}, \tilde{y})\right)\right).
\end{aligned}$$

Now from (26) and (27), we see that

$$|\tilde{\bar{u}} - \tilde{u}\left(t, x, y, h(t, x, y)\right)| = O\left(\beta\right),$$

and

$$|\tilde{\bar{v}} - \tilde{v}\left(t, x, y, h(t, x, y)\right)| = O\left(\beta\right).$$

Hence, the vorticity of mean horizontal velocity can be seen to be of the order of β. As a consequence, one can also use the following approximate formula for the horizontal velocity field at the free surface in terms of the average horizontal velocity:

$$\tilde{\mathbf{u}}_H(\tilde{\mathbf{x}}, \tilde{h}, \tilde{t}) = \tilde{\bar{\mathbf{u}}} - \beta\frac{\tilde{h}^2}{3}\tilde{\nabla}_H\left(\tilde{\nabla}_H \cdot \tilde{\bar{\mathbf{u}}}\right) + \mathcal{O}(\beta^2).$$

References

[1] A. Ali and H. Kalisch, *Mechanical balance laws for Boussinesq models of surface water waves*, J. Nonlinear Sci. **22** (2012), 371–398.

[2] A. Ali and H. Kalisch, *On the formulation of mass, momentum and energy conservation in the KdV equation*, Acta Appl. Math. **133** (2014), 113–131.

[3] C. Bardos, D. Lannes, *Mathematics for 2d interfaces*, Singularities in Mechanics: Formation, Propagation and Microscopic Description, Panoramas et syntheses **38** (2012)

[4] R. Barros, S. Gavrilyuk and V. Teshukov, *Dispersive nonlinear waves in two-layer flows with free surface. I. Model derivation and general properties*, Studies in Applied Mathematics, **119** (2007), 191–211.

[5] E. Barthélemy, *Nonlinear shallow water theories for coastal waves*, Surv. Geophys. **25** (2004), 315–337.

[6] T.B. Benjamin and P.J. Olver, *Hamiltonian structure, symmetries and conservation laws for water waves* J. Fluid Mech. **125** (1982), 137–185.

[7] M. Bjørkavåg and H. Kalisch. *Wave breaking in Boussinesq models for undular bores*, Phys. Lett. A **375** (2011), 1570–1578.

[8] P. Bonneton, E. Barthelemy, F. Chazel, R. Cienfuegos, D. Lannes, F. Marche and M. Tissier, *Recent advances in Serre-Green-Naghdi modelling for wave transformation, breaking and runup processes*, European J. Mech. B **30** (2011), 589–597.

[9] P. Bonneton, F. Chazel, D. Lannes, F. Marche and M. Tissier, *A splitting approach for the fully nonlinear and weakly dispersive Green-Naghdi model* J. Comput. Phys. **230** (2011), 1479–1498.

[10] H. Borluk and H. Kalisch, *Particle dynamics in the KdV approximation*, Wave Motion **49** (2012), 691–709.

[11] R. Camassa, D.D. Holm, and C.D. Levermore, *Long-time effects of bottom topography in shallow water*, Phys. D **98** (1996), 258–286.

[12] J.D. Carter and R. Cienfuegos, *The kinematics and stability of solitary and cnoidal wave solutions of the Serre equations*, Eur. J. Mech. B/Fluids **30** (2011), 259–268.

[13] A. Castro and D. Lannes, *Fully nonlinear long-waves models in presence of vorticity*, J. Fluid Mech. **759** (2014), 642–675.

[14] A. Castro and D. Lannes, *Well-posedness and shallow water stability for a new Hamiltonian formulation of the water waves equations with vorticity*, http://arxiv.org/abs/1402.0464, to appear in Indiana University Math. Journal

[15] F. Chazel, D. Lannes and F. Marche, *Numerical simulation of strongly nonlinear and dispersive waves using a Green-Naghdi model*, J. Sci. Comput. **48** (2011), 105–116.

[16] W. Choi and R. Camassa, *Fully nonlinear internal waves in a two-fluid system*, J. Fluid Mech. **396** (1999), 1–36.

[17] R. Cienfuegos, E. Barthélemy, P. Bonneton, *A fourth-order compact finite volume scheme for fully nonlinear and weakly dispersive Boussinesq-type equations, Part I: model development and analysis*, Internat. J. Numer. Methods Fluids **51** (2006), 1217–1253.

[18] F. Dias, P. Milewski, *On the fully-nonlinear shallow-water generalized Serre equations*, Phys. Lett. A **374** (2010), 1049–1053.

[19] D. Dutykh, D. Clamond, P. Milewski, and D. Mitsotakis, *Finite volume and pseudo-spectral schemes for the fully nonlinear 1D Serre equations*, arXiv: 1104.4456v4 [Physics.flu-dyn] 13 Feb 2013.

[20] G.A. El, R.H.J. Grimshaw and N.F. Smyth, *Unsteady undular bores in fully nonlinear shallow-water theory*, Phys. Fluids **18** (2006), 027104.

[21] S. Gavrilyuk and V. Teshukov, *Generalized vorticity for bubbly liquid and dispersive shallow water equations*, Continuum Mechanics and Thermodynamics **13** (2001), 365–382.

[22] A.E. Green, N. Laws, and P.M. Naghdi, *On the theory of water waves*, Proc. R. Soc. Lond. **A 338** (1974), 43–55.

[23] A. E. Green and P. M. Naghdi, *A derivation of equations for wave propagation in water of variable depth*, J. Fluid Mech. **78** (1976), 237–246.

[24] S. Israwi, *Derivation and analysis of a new 2D Green-Naghdi system*, Nonlinearity **23** (2010), 2889–2904.

[25] S. Israwi, *Large time existence for 1d Green-Naghdi equations*, Nonlinear Analysis TMA **74** (2011), 81–93.

[26] H. Kalisch, Z. Khorsand and D. Mitsotakis, *Mechanical balance laws for the Green-Naghdi system and applications to undular bores, shoaling and wave loading*, in preparation.

[27] Z. Khorsand, *Particle trajectories in the Serre equations*, Appl. Math. Comp. **230** (2014), 35–42.

[28] J.W. Kim, K.J. Bai, R.C. Ertekin and W.C. Webster, *A* derivation of the Green-Naghdi equations for irrotational flows, J. Engrg. Math. **40** (2001), 17–42.

[29] D. Lannes, The Water Waves Problem. Mathematical Surveys and Monographs, vol. **188** (Amer. Math. Soc., Providence, 2013).

[30] D. Lannes and P. Bonneton, *Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation*, Phys. Fluids **21** (2009), 016601.

[31] M. Li, P. Guyenne, F. Li and L. Xu, *High order well-balanced CDG-FE methods for shallow water waves by a Green-Naghdi model*, J. Comp. Physics **257** (2014), 169–192.

[32] Y.A. Li, *Linear stability of solitary waves of the Green-Naghdi equations*, Comm. Pure Appl. Math. **54** (2001), 501–536.

[33] Y.A. Li, *A shallow-water approximation to the full water wave problem*, Comm. Pure Appl. Math. **59** (2006), 1225–1285.

[34] N. Makarenko, A second long-wave approximation in the Cauchy-Poisson problem, Dynamics of Continuous Media, v. 77 (1986) , pp. 56-72 (in Russian).

[35] O. Le Métayer, S. Gavrilyuk, and S. Hank, *A numerical scheme for the Green-Naghdi model*, J. Comp. Phys. **229** (2010), 2034–2045.

[36] D.H. Peregrine, *Long waves on beaches*, Fluid Dynamics Transactions **10** (1980), 77–111.

[37] J. Miles and R. Salmon, *Weakly dispersive nonlinear gravity waves*, J. Fluid Mechanics **157** (1985), 519–531.

[38] J.R. Percival, C.J. Cotter and D.D. Holm, *A Euler-Poincaré framework for the multilayer Green-Naghdi equations*, J. Physics A : Mathematical and General **41** (2008), 344018.

[39] J.-C. Saut and L. Xu, *Well-posedness on large time for a modified full dispersion system of surface waves* J. Math. Phys. **53** (2012), 115606, 12 pp.

[40] F. Serre, *Contribution à l'étude des écoulements permanents et variables dans les canaux*, La Houille Blanche **8** (1953), 374–388.

[41] C.H. Su and C.S. Gardner *Korteweg-de Vries equation and generalizations, III. Derivation of the Korteweg-de Vries equation and Burgers equation*, J. Math. Phys. **10** (1969), 536–539.

[42] V.E. Zakharov, *Benney equations and quasiclassical approximation in the method of the inverse problem*, Functional Analysis and its applications **14** (1980), 89–98.

[43] Y. Zhang, A. B. Kennedy, N. Panda, C. Dawson, and J. J. Westerink, *Boussinesq-Green-Naghdi rotational water wave theory*, Coast. Eng. **73** (2013), 13–27.